Example 2: Evaluate the integral: $\int \frac{x^2-1}{x(x^2+1)^2}\,dx = I$

We want to apply partial fractions:

→ leading term on the denom is one ✓

→ $\deg(\text{num}) = 2$, $\deg(\text{denom}) = \cancel{3}\ 5$
(No polynomial long division)

→ cannot factor denom.

$x(x^2+1)^2$ any further

$\longrightarrow$ directly apply the partial fractions procedure:

$$\frac{x^2-1}{x(x^2+1)^2} = \frac{A}{x} + \underbrace{\frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}}_{(x^2+1)^J=(x^2+1)^2 \longleftrightarrow J=2} \quad (*)$$

$x = x^k \longleftrightarrow k=1$

→ Multiply both sides of (*) by the common denominator $x(x^2+1)^2$:

$$x^2-1 = A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x \qquad (**)$$

→ Now what we need to do is to plug in specific values of x to solve for A, B, C, D, E

good values of x to choose:

$x = 0, \pm 1, \pm 2$ into $(**)$

with $x=0$: $-1 = A + 0 + 0 \iff A = -1$

with $x=+1$: $0 = 4A + 2B + 2C + D + E$

$\iff 4 = 2B + 2C + D + E \quad (2)$

$$x^2-1 = A(x^2+1)^2 + (Bx+C)(x^2+1)x + (Dx+E)x \qquad (**)$$

with $x=-1$: $0 = 4A + 2B - 2C + D - E$

$$4 = 2B - 2C + D - E \qquad (3)$$

$$(2)+(3) \longrightarrow 8 = 4B + 2D$$

$$\longleftrightarrow 4 = 2B + D$$

$(2)-(3) \longrightarrow 0 = 4C + 2E$

$\longleftrightarrow 0 = 2C + E$

<u>with $x=+2$:</u> $3 = 25A + 20B + 10C + 4D + 2E$

$\longleftrightarrow 14 = 10B + 5C + 2D + E \quad (4)$

<u>with $x=-2$:</u> $3 = 25A + 20B - 10C + 4D - 2E$

$\Leftrightarrow \quad 14 = 10B - 5C + 2D - E \qquad (5)$

$(4)+(5) \rightarrow 28 = 20B + 4D$

$\Leftrightarrow \quad 7 = 5B + D$

$(4)-(5) \rightarrow 0 = 10C + 4E$

$\Leftrightarrow \quad 0 = 5C + 2E$

$\rightarrow$ Now we have 4 eqns. for 4 unknowns:

$$4 = 2B + D$$
$$7 = 5B + D$$
$$\rightarrow -3 = -3B$$
$$\rightarrow B = 1$$

$$4 = 2(1) + D \rightarrow D = 2$$

$$0 = 2C + E$$
$$0 = 5C + 2E$$
$$\rightarrow -C = 0$$
$$C = 0$$
$$\rightarrow E = 0$$

So, in total: $A=-1$, $B=1$, $C=0$,
$D=2$, $E=0$

$$\frac{x^2-1}{x(x^2+1)^2}=\frac{A}{x}+\underbrace{\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2}}\quad (*)$$

$$I=-\int\frac{dx}{x}+\int\frac{x}{x^2+1}dx+\int\frac{2x\,dx}{(x^2+1)^2}$$

$I_1 \qquad I_2 \qquad I_3$

$I_1 = -\ln|x| + C_1$

I_2: u-sub: $u = x^2$, $du = 2x\,dx$

$$I_2 = \frac{1}{2}\int \frac{du}{u+1} = \frac{1}{2}\ln|u+1| + C_2$$

$$= \frac{1}{2}\ln|x^2+1| + C_2$$

I_3: take the same u-sub:

$$I_3 = \int \frac{du}{u^2} = -\frac{1}{u} + C_3$$
$$= -\frac{1}{x^2} + C_3$$

In total:

$$I = -\ln|x| + \frac{1}{2}\ln|x^2+1| - \frac{1}{x^2} + C$$

Example 3: Evaluate the integral: $\int \frac{2x-1}{x^2(x-2)^2}\,dx$

→ work through this example on your own

Example 4: Evaluate the definite integral: $\int_{\pi/3}^{\pi/2} \frac{\sin\vartheta}{\cos^2\vartheta+\cos\vartheta-2}\,d\vartheta = I$

$\rightarrow$ what to do first? (u-sub)

$u=\cos\theta$, $du = -\sin(\theta)d\theta$

$\cos(\pi/2) = 0$

$\cos(\pi/3) = 1/2$

$\int_a^b f(x)dx = -\int_b^a f(x)dx$

$$I = \int_0^{1/2} \frac{du}{u^2+u-2}$$

$$n^2+n-2=(n+2)(n-1)$$

$\longrightarrow$ apply partial fractions:

$$\frac{1}{(n-2)(n+1)} = \frac{A}{n+2} + \frac{B}{n-1} \quad (*)$$

$$1 = A(n-1) + B(n+2)$$

good values to choose: $n=+1, -2$

with $n=+1$: $1 = 3B \longleftrightarrow B = 1/3$

With $u=-2$: $1=-3A \longleftrightarrow A=-1/3$ Plug in to *

$$\to I=-\frac{1}{3}\int_0^{1/2}\frac{du}{u-2}+\frac{1}{3}\int_0^{1/2}\frac{du}{u+1}$$

$$=-\frac{1}{3}\left(\ln|u-2|-\ln|u+1|\right)\Big|_0^{1/2}$$

$$=-\frac{1}{3}\left(\ln\left|\frac{u-2}{u+1}\right|\right)\Big|_0^{1/2}$$

$$= -\frac{1}{3}\underbrace{\ln\left|\frac{3/2}{3/2}\right|}_{\ln(1)=0} + \frac{1}{3}\ln\left|\frac{2}{1}\right|$$

$$= +\frac{\ln(2)}{3}$$

Suppose we used the method from the pre-lecture video to solve for A, B instead:

$\longrightarrow$ cross multiplying gives:

$$1 = A(n-1) + B(n-2)$$

$$\Longleftrightarrow (A+B)\cdot n + (2B-A) = 0\cdot n + 1$$

$$\Longleftrightarrow \quad A+B=0 \text{ or } A=-B \quad (1)$$

$$2B-A=1 \quad (2)$$

plug (1) into (2) to get:

$$2B + B = 1 \Longleftrightarrow B = 1/3 \quad (3)$$

Plug (3) into (1) to get: $A = -1/3$

(same answers as before ☺)

Review Question: Which of the following integrals would you evaluate using partial fractions? Why?

$\sim$ (A) $\displaystyle\int \frac{x}{4-x^2}\,dx$

✓ (B) $\displaystyle\int \frac{x^2-2}{x^2(x-3)^2}\,dx$

✗ (C) $\displaystyle\int \frac{x}{1+x^4}\,dx$

✓ (D) $\displaystyle\int \frac{x+1}{x^3+6x^2+9x}\,dx$

(A) either works: u-sub with $u=x^2$

(B) Yes!

(C) use u-sub: $u=x^2$, $du=2x\,dx$

$$I=\frac{1}{2}\int \frac{du}{1+u^2}$$

Math 1552

Section 4.5:
L'Hopital's Rule

Math 1552 lecture slides adapted from the course materials
By Klara Grodzinsky (GA Tech, *School of Mathematics*, Summer 2021)

Today’s Learning Goals

- Understand which forms are indeterminate
- Apply L’Hopital’s Rule to evaluate limits
- Rewrite limits in forms appropriate to applying L’Hopital’s Rule

Indeterminate Forms

$$\frac{0}{0}, \frac{\infty}{\infty}$$

can directly apply L'Hopital's rule

$$1^{\infty}, 0^{0}, \infty^{0}$$

$$0 \cdot \infty, \infty - \infty$$

Which of the following limits
does NOT contain an indeterminate form? Why?

$x+1 \to \infty$
$3x \to \infty$

X A. $L = \lim_{x\to\infty} (x+1)^{3x}$ ∞^{∞}, or $\ln(L) = \lim_{x\to\infty} 3x \ln(x+1)$

$= \infty \cdot \infty$

√ B. $\lim_{x\to 0^+} x^{6x} \to 0^0$

√ C. $\lim_{x\to\infty} x^2 e^{-x}$

√ D. $\lim_{x\to 0^+} (\cos x)^{\frac{1}{x}}$

$\cos(0) = 1$

$\frac{1}{x} \to \infty$

1^{∞}

(c) $e^{-x} \to 0$

$x^2 \to \infty$

$\Rightarrow 0 \cdot \infty$

L'Hopital's Rule

Let *f* and *g* be two functions. Then IF:

a) *f* and *g* are differentiable,

b) $\dfrac{f(x)}{g(x)}$ has the indeterminate form of

$$\frac{0}{0} \quad \text{OR} \quad \frac{\infty}{\infty}$$

i-portant

c) $\lim_{x\to c} \dfrac{f'(x)}{g'(x)} = L$, exists

THEN: $$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)} = L$$

Example 1.1: Use L'Hopital's rule to evaluate the following limit.

$$L = \lim_{x\to\infty} \frac{e^x + x^2}{e^x + x} \quad \begin{matrix} f \\ g \end{matrix}$$

$\frac{\infty}{\infty}$ as our indet. form

$\longrightarrow$ apply L'Hop's rule

$$L = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

$$= \lim_{x\to\infty} \frac{e^x + 2x}{e^x + 1}$$

$\frac{\infty}{\infty}$ again, apply L'Hop. a second time

$$= \lim_{x \to \infty} \frac{e^x + 2}{e^x}$$

$$= \lim_{x \to \infty} \left(1 + 2e^{-x}\right) = 1$$